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Momentum space trigonometric Rosen–Morse potential

C B Compean and M Kirchbach

Instituto de Fisica, UASLP, Av. Manuel Nava 6, Zona Universitaria, San Luis Potosi, SLP 78290, Mexico

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Abstract
We transform the trigonometric S wave Rosen–Morse potential to momentum space by employing its property of being a harmonic angular function on the three-dimensional hypersphere S^3.

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Momentum space potentials obtained as Fourier transforms of central potentials are of interest in a variety of physics problems ranging from condensed matter to particle physics. They can be viewed as instantaneous propagators of the fields mediating the respective interactions and are especially important in Faddeev few-body calculations which are more efficiently carried out in momentum than in position space. Unfortunately, the power potentials of wide spread such as linear and harmonic oscillator interactions do not have well-defined Fourier integrals [1], the inverse distance potential being the most prominent exception.

We here make the case that the S wave trigonometric Rosen–Morse potential when considered as an angular function on the three-dimensional (3D) surface of constant positive curvature, the S^3 hypersphere, allows for a momentum space transform that can be cast in a closed form.

The $\cot + \csc^2$ interaction, known as the trigonometric Rosen–Morse potential and managed by SUSYQM [2], was in fact invented by Schrödinger in [3] and was originally introduced as an angular function on a three-dimensional (3D) surface of constant positive curvature, the hypersphere S^3 embedded in a flat Euclidean space of four dimensions, E_4. Up to additive constants, it takes the form

$$V_{\mathrm{RM}}(\chi) = -2B\cot\chi + \frac{\hbar^2}{2\mu d^2} l(l+1)\csc^2\chi. \qquad (1)$$

Here l is the value of the 3D angular momentum, d is a matching length constant, while χ is the second polar angle in E_4. In choosing the parameterization, $\chi = \frac{r}{R}$, for the angular variable, where R is the constant radius of S^3, while $r = R\chi$ is the length of the arc on the hyperspherical surface, $V_{\rm RM}(\chi)$ is usually given the form of a potential in a 3D space, though not a flat one. The 3D flat Euclidean space, E_3, embedded in E_4 is described in terms of a radius vector of an absolute value, $|\mathbf{r}|$, defined as $|\mathbf{r}| = R\sin\chi$. Therefore, the χ parametrization corresponding to the correct 3D flat space embedded in E_4 is $\chi = \sin^{-1}\frac{|\mathbf{r}|}{R}$.

The nature of the space, flat versus curved, is of minor importance for the energy spectrum and the wavefunctions, and reduces to the interpretation of R. In the flat space R is viewed as some matching length parameter, while S^3 puts it on the firmer ground of a parameter encoding the curvature. Yet, regarding integral transformations such as Fourier transforms to momentum space, the nature of the space acquires significance through the definition of the integration volume. Trying to use the flat space E_3 integral volume and a 3D plane wave to Fourier transform $V_{\rm RM}\big(\chi = \frac{r}{R}\big)$ as a function of the arc, r, is inconsistent and leads to a divergent Fourier integral. Considering instead $V_{\rm RM}$ as a function of the radius vector of the correct flat E_3 space, underlying E_4, allows for a Fourier transform that can be taken in a closed form. Below we calculate the 4D Fourier transform of $V_{\rm RM}\big(\chi = \sin^{-1}\frac{|\mathbf{r}|}{R}\big)$ to momentum space.

Angular potentials in extra dimensions are important because they allow us to replace complicated many-body problems in flat space by effective two-body systems on curved spaces with the curvature parameter absorbing the many-body effects. This is a well-known technique which has been applied in several physics problems ranging from plasma to instanton physics [4–6]. Specifically, the trigonometrical Rosen–Morse potential has found an interesting application in the physics of strongly interacting elementary particles [7]. It has been shown to act as the exactly solvable extension to the quark confinement potential [8] obtained in solving the equation of quantum chromodynamics by the technique of simulations on a lattice, an observation reported in [9]. As another relevant application of the same potential, we wish to mention its use in the theory of quantum dots [10].

Treating the interaction under discussion as an angular function on S^3 is possible because of its $SO(4)$ symmetry. The latter is best understood by observing that the Schrödinger equation with the $\cot + \csc^2$ potential is closely related to the eigenvalue problem of the 4D angular momentum on S^3. Throughout the paper, we consider ordinary Euclidean flat space, E_3, embedded in a 4D Euclidean space, E_4, and parametrize the 3D spherical surface S^3 as $x_4^2 + \mathbf{r}^2 = R^2$ with $x_4 = R\cos\chi$, and $|\mathbf{r}| = R\sin\chi$. Here, R is the constant hyper-radius of S^3. The angular part, $\widehat{\underline{\sqcap\sqcap}}$, of the 4D Laplace–Beltrami operator is proportional to the operator of the squared 4D angular momentum, $\mathcal{K}^2$, and is given by

$$\widehat{\underline{\sqcap\sqcap}} = \left[\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} - \frac{\mathbf{L}^2(\theta,\varphi)}{\sin^2\chi}\right] = -\kappa\mathcal{K}^2, \qquad \kappa = \frac{1}{R^2}. \tag{2}$$

Here $\mathbf{L}^2(\theta,\varphi)$ is the standard 3D orbital angular momentum operator in E_3, the ordinary position space[1], χ is the second polar angle in E_4, $\chi \in [0,\pi]$, while κ stands for the constant curvature. Consequently, the Schrödinger equation on S^3 becomes

$$\left[\frac{\hbar^2}{2\mu}\kappa\mathcal{K}^2 - E(\kappa)\right]\psi(\chi,\kappa) = 0, \tag{3}$$

where μ stands for the reduced mass. The $\mathcal{K}^2$ eigenvalue problem reads [11]

$$\mathcal{K}^2|Klm\rangle = K(K+2)|Klm\rangle, \qquad |Klm\rangle \in \left(\frac{K}{2},\frac{K}{2}\right). \tag{4}$$

The $|Klm\rangle$-levels belong to irreducible $SO(4)$ representations of the type $\big(\frac{K}{2},\frac{K}{2}\big)$, and the quantum numbers, $K, l,$ and m define the eigenvalues of the respective four-, three- and

[1] The analog on the 2D sphere, S^2, of constant radius $|\mathbf{r}| = a$, is the well-known relation $\vec{\nabla}^2 = -\frac{1}{a^2}\mathbf{L}^2$.

two-dimensional angular momentum operators upon the state. These quantum numbers correspond to the $SO(4) \supset SO(3) \supset SO(2)$ reduction chain and satisfy the branching rules, $l = 0, 1, 2, \ldots, K$, and $m = -l, \ldots, +l$. Multiplying equation (3) by $(-\sin^2\chi)$ and changing the variable to $\psi(\chi, \kappa) = \sin\chi \mathcal{S}(\chi, \kappa)$ results in the following Schrödinger equation:

$$\left[-\kappa\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + U_l(\chi, \kappa)\right]\mathcal{S}(\chi, \kappa) = E(\kappa)\mathcal{S}(\chi, \kappa), \qquad U_l(\chi, \kappa) = \kappa\frac{\hbar^2}{2\mu}l(l+1)\csc^2\chi, \tag{5}$$

with $U_l(\chi, \kappa)$ now having the meaning of a centrifugal barrier on S^3. As a different interpretation of equations (3) and (5), one can say that the $\csc^2$ potential, in representing the centrifugal barrier on the 3D hypersphere, has $SO(4)$ as a potential algebra. An important observation is that the potential algebra $SO(4)$ remains unaltered upon adding to the $\csc^2$ term the harmonic fuction, $\cot\chi$. This is visible from the fact that the energy continues being a function of the $\mathcal{K}^2$ eigenvalues $K(K+2)$ alone which translate into the principal quantum number n used in [2] as $n = K + 1$. In effect, the $SO(4)$ symmetry of the $\cot + \csc^2$ interaction allows us to consider it as an angular function on S^3, a circumstance that will substantially facilitate its transformation to momentum space. We here adopt the following parametrization of the trigonometric Rosen–Morse potential as a function of the second polar angle, χ, on S^3, and the curvature:

$$\mathcal{V}(\chi) = -2G\sqrt{\kappa}\cot\chi + \kappa\frac{\hbar^2}{2\mu}\frac{l(l+1)}{\sin^2\chi}. \tag{6}$$

In Cartesian coordinates, the $\cot\chi$ term equals $\frac{x_4}{|\mathbf{r}|}$, and stands in fact for two potentials distinct by a sign and describing interactions on the respective Northern and Southern hemispheres. Correspondingly, their respective Fourier transforms to momentum space become

$$\begin{aligned} 4\pi\,\Pi(|\mathbf{q}|) = {}& -2G\sqrt{\kappa}\frac{2\mu}{\hbar^2}\int_0^\infty \mathrm{d}|x||x|^3\delta(|x| - R)\int_0^{2\pi}\mathrm{d}\varphi\int_0^\pi \mathrm{d}\theta\,\sin\theta \\ & \times \int_{0/\frac{\pi}{2}}^{\frac{\pi}{2}/\pi}\mathrm{d}\chi\,\sin^2\chi\,\mathrm{e}^{\mathrm{i}|\mathbf{q}|\,\frac{\sin\chi}{\sqrt{\kappa}}\,|\cos\theta}\cot\chi, \end{aligned} \tag{7}$$

where the $\delta(|x| - R)$ function restricts E_4 to S^3. Here, the 4D plane wave has been evaluated in reference to a z-axis chosen along the momentum vector (a choice justified in elastic scattering[3]) and a position vector of the confined particle having in general a non-zero projection on the extra dimension axis in E_4:

$$\mathrm{e}^{\mathrm{i}q\cdot x} = \mathrm{e}^{\mathrm{i}|\mathbf{q}||\mathbf{r}|\cos\theta} = \mathrm{e}^{\mathrm{i}|\mathbf{q}|\,\frac{\sin\chi}{\sqrt{\kappa}}\,\cos\theta}, \qquad |\mathbf{r}| = R\sin\chi = \frac{\sin\chi}{\sqrt{\kappa}}. \tag{8}$$

On S^3 one has to distinguish between two types of momentum space potentials. The first one, displayed in figure 1, goes with $\chi \in \left[0, \frac{\pi}{2}\right]$, corresponds to a positive x_4, and describes an

[2] Harmonic angular functions in E_4 are $\mathcal{K}^2$ eigenfunctions belonging to zero eigenvalues The function $\cot\chi$ of the second polar angle is such a quantity, and the counterpart on S^3 to the harmonic S^2 function, $\ln\tan\frac{\theta}{2}$, of the first polar angle which satisfies $\nabla^2\ln\tan\frac{\theta}{2} = 0$. The general mathematical theory of angular potentials and related harmonic functions has been developed by Gabov in [12] and references therein.

[3] A consistent definition of the E_4 plane wave would require a Euclidean q vector. However, for elastic scattering processes, of zero energy transfer, where $q_0 = 0$, the q vector can be chosen to lie entirely in E_3, and be identified with the physical spacelike momentum transfer.

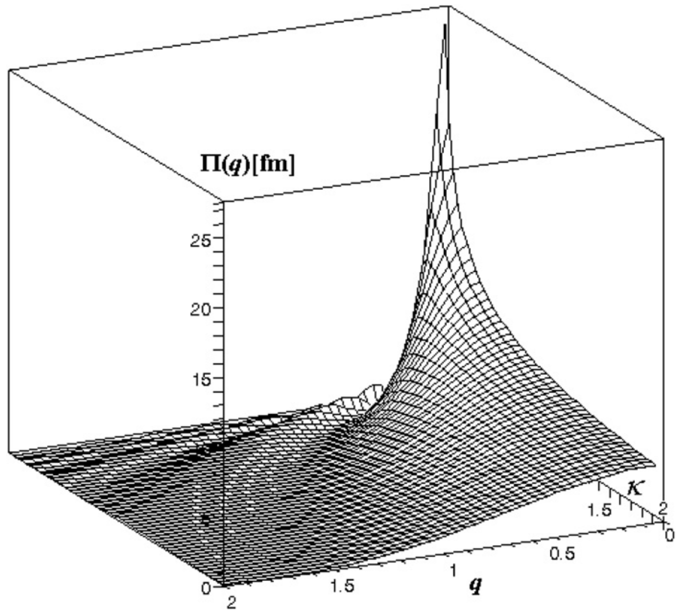

Figure 1. The curvature dependence of the momentum space potential in the Northern hemisphere. The Southern part appears mirrored with respect to the horizontal plane. Both potentials approach zero at infinity.

increasing $|\mathbf{r}|$, while the second refers to $\chi \in \left[\frac{\pi}{2}, \pi\right]$, a negative x_4, and describes a decreasing $|\mathbf{r}|$. The first type refers to the Northern hemisphere and reads

$$\Pi(|\mathbf{q}|) = c\frac{2\sin^2\frac{|\mathbf{q}|}{2\hbar\sqrt{\kappa}}}{\left(\frac{|\mathbf{q}|}{\hbar\sqrt{\kappa}}\right)^2}, \qquad c = 2G\frac{2\mu}{\hbar^2\kappa}. \tag{9}$$

It is increasing in the infrared, finite at origin, and approaches asymptotically the Coulomb propagator in the ultraviolet. Such a type of behavior is required, for example, in the description of confinement phenomena [13]. If one had treated instead the cot potential as a flat space interaction, the 3D Fourier integral would have been divergent [9] and one would have been forced to introduce a π range correlation function in order to get it finite as we did in [14].

In summary, one of the virtues of the curvature aspect of the cot interaction is that its S^3 Fourier transform comes out well defined.

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